UNIVERSITÄT ZU LÜBECK
INSTITUTE FOR ROBOTICS
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Humanoid Robotics
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Exercise sheet 3 - Control Theory

Please prepare the following exercises for the upcoming tutorial.

Task 1: Closed-Loop PID Control

In this exercise we will build a Closed-Loop PID Controller from the scratch. Thus, no previous knowledge about control design will be required. As an example throughout this exercise we will use a simple spring-damper model depicted in Figure 1 given the following parameters

$$m = 1, \qquad b = 0.7, \qquad k = 1. \tag{1}$$

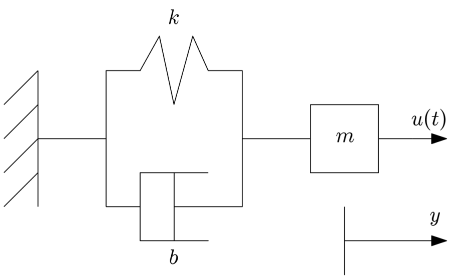

Figure 1 Spring-damper model

a) Derive the differential equation for the spring-damper model as depicted in Figure 1!

b) Rewrite the differential equation from (a) into a system of differential equations with order 1! Please use the standard matrix notation as presented in exercise (1).

c) Give the definition of the standard linear State-Space-Model and transform the system of differential equation from (b) into such a model!

d) State the mathematical description of the Laplace Transform! Explain what this transformation does and transform the following general from of a differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}y(t) + a_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}y(t) + \cdots + a_1\dot{y}(t) + a_0 y(t) = b_m\frac{\mathrm{d}^m}{\mathrm{d}t^m}u(t) + \cdots + b_1\dot{u}(t) + b_0 u(t). \tag{2}$$

Define the transfer function for the transformed equation! Transform the differential equation generated in (a)!

Hint: You find a table with Laplace Transformation in the appendix from http://tutorial.math.lamar.edu.

e) Transform the State-Space Model from (c) with the Laplace Transformation and state the transfer function! Transform the State-Space Model generated in (c)!

f) State the final and initial value theorems! Use the final value theorem to calculate the answer of the system given in Figure 1 to a unit step response for $t \to \infty$.

g) Define the poles of the transfer function and calculate them for the in (d) or (e) calculated transfer function for the spring-damper model! Explain which condition the poles have to full-fill in order to guarantee stability for a given system!

h) Draw a sketch for a Closed-Loop System and derive the Close-Loop transfer function!

i) Draw a sketch for a Closed-Loop Control System and derive the transfer function!

j) Define the general framework of a PID controller in time- and frequency domain! Design three controller (P, PD, PID) and use them to control the above presented spring-damper model if a unit disturbance step is added to the system!

Task 2: LQR

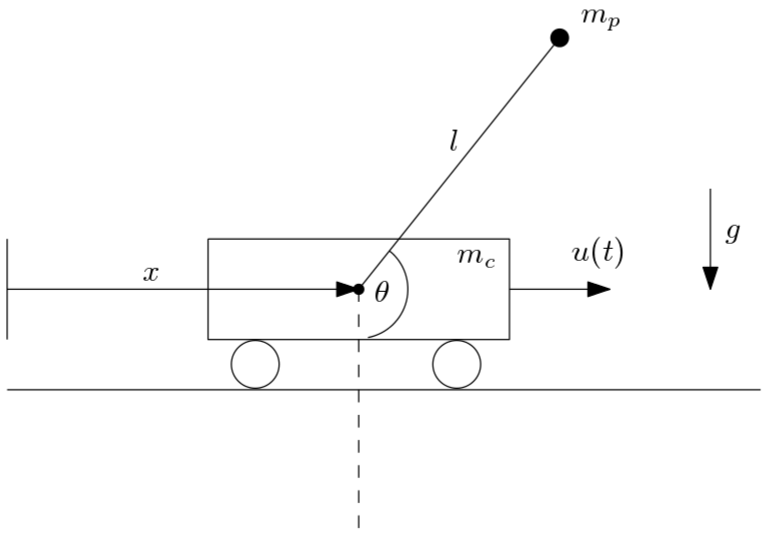

Figure 2 Cart-Pole

a) Derive the State-Space System for the in exercise 1 presented cart-pole. Since we are using a Linear-Quadratic-Regulator, we have to linearize the system around $\theta = \pi - \phi$. Please add also an input term $u(t)$ into the equation as shown in Figure 2.

b) Check the stability and controllability of the system!

c) Explain the general idea of an LQR controller!

d) Design a LQR controller which can steer the car 0.2 metre to the right by holding the following design criteria:

  - Settling time for x and ϕ less than 5 seconds
  - Rise time for x of less than 0.5 seconds
  - Pendulum angle ϕ never more than 20 degrees
  - Steady-State error less than 2 percent for x and y

Graded Assignment 03: LQR and PID Control

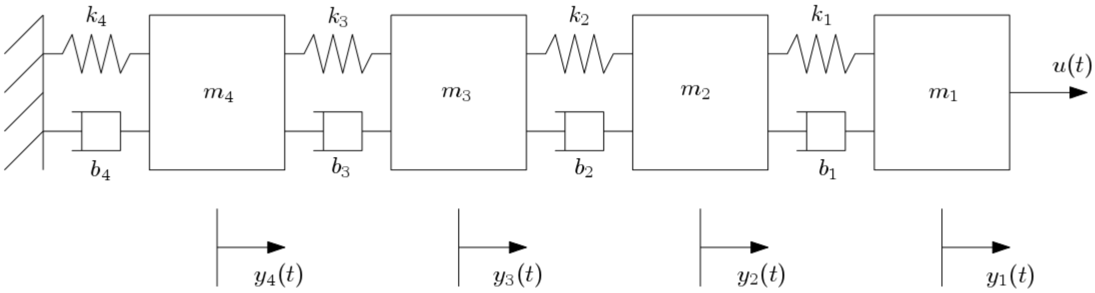

Figure 3 Spring-Damper System

In this assignment you have to control the in Figure 3 shown system given the following parameters

$$\begin{array}{cccc} m_1 = 1 & m_2 = 0.5 & m_3 = 2 & m_4 = 0.5 \\ k_1 = 1 & k_2 = 0.5 & k_3 = 0.5 & k_4 = 1 \\ b_1 = 0.2 & b_2 = 0.5 & b_3 = 0.1 & b_4 = 0.3 \end{array}. \tag{3}$$

Therefore you first have to derive the system of differential equations. Based on this system you can generate a State-Space Model. Applying a unit step force $u(t) = \sigma(t)$ should lead to the result depicted in Figure 4. Your mission should you choose to accept it, is to design a closed-loop feedback controller which improves the behavior of the system and leads to $\lim_{t\to\infty} y_1(t) = 1$. Please design different controller, but at least one with a Linear-Quadratic Regulator (LQR) and one with a PID Controller. It may also be interesting to analyze the different impact of the P, D and I part from the PID controller. A possible solution for a controlled system (LQR) is shown in Figure 5. Please analyze, before you start to design your controller, if the uncontrolled system is stable and if it is controllable.

In order to pass the assignment you have to write a Matlab code which generates the system presented above. The system has to be controlled with both, LQR and PID control. The Matlab code has to be executable using a main file. Figures which show the results of your controllers should be opened by the Matlab script. In addition, 2-4 pages as a PDF document have to be submitted. In this document the principles you used as well as your results shall be presented. The maximum number of points reachable is 12.

The submission deadline for this assignment is June 22, 2018, 10am. Please send your submission as a Zip data named RO5300_TeamNumber to Nils.Rottmann@rob.uni-luebeck.de with the subject RO5300_TeamNumber. Other submissions will not be considered.

If you have any problems with the assignment feel free to get in touch. You will find me in Building 64, Room 85.

You can earn up to 6 bonus points in this assignment, three for each of the following additional tasks:

- Program the LQR algorithm by yourself, so do not use the Matlab function *lqr*. In addition write one page about the LQR method.
- Use an observer in order to estimate the full state of the system. In addition write one page about observability and the state estimation.

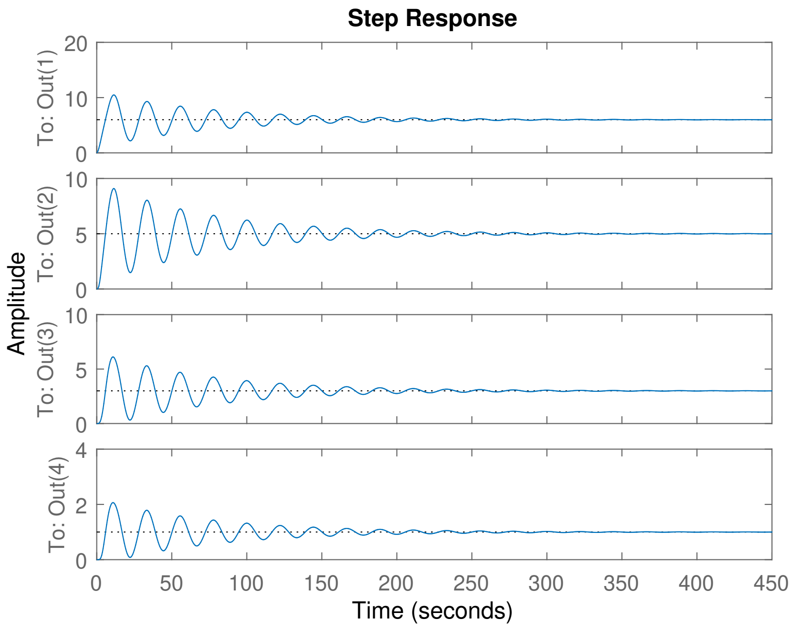

Figure 4 Step Response Spring-Damper System

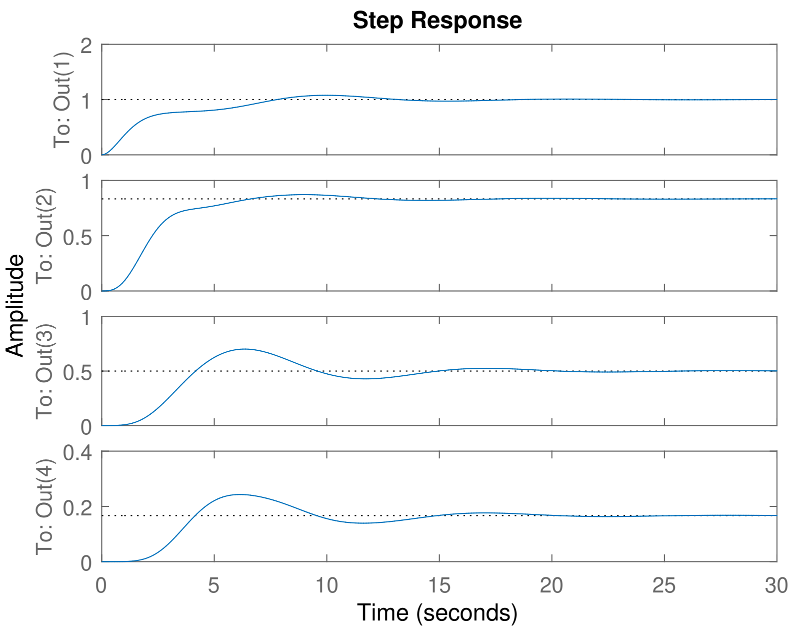

Figure 5 Step Response of controlled Spring-Damper System

Table of Laplace Transforms

$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. $\mathbf{e}^{at}$	$\frac{1}{s-a}$
3. $t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4. $t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6. $t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10. $t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11. $\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12. $\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13. $\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14. $\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15. $\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17. $\sinh(at)$	$\frac{a}{s^2-a^2}$	18. $\cosh(at)$	$\frac{s}{s^2-a^2}$
19. $\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20. $\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21. $\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22. $\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23. $t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25. $u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26. $\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27. $u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28. $u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29. $\mathbf{e}^{ct}f(t)$	$F(s-c)$	30. $t^nf(t),\quad n=1,2,3,\ldots$	$(-1)^nF^{(n)}(s)$
31. $\frac{1}{t}f(t)$	$\int_s^{\infty}F(u)\,du$	32. $\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34. $f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35. $f'(t)$	$sF(s)-f(0)$	36. $f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37. $f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$		

Table Notes

1. This list is not a complete listing of Laplace transforms and only contains some of the more commonly used Laplace transforms and formulas.

2. Recall the definition of hyperbolic functions.

$$\cosh(t) = \frac{\mathbf{e}^t + \mathbf{e}^{-t}}{2} \qquad \sinh(t) = \frac{\mathbf{e}^t - \mathbf{e}^{-t}}{2}$$

3. Be careful when using "normal" trig function vs. hyperbolic functions. The only difference in the formulas is the "+ a^2" for the "normal" trig functions becomes a "- a^2" for the hyperbolic functions!

4. Formula #4 uses the Gamma function which is defined as

$$\Gamma(t) = \int_0^\infty \mathbf{e}^{-x} x^{t-1}\, dx$$

If n is a positive integer then,

$$\Gamma(n+1) = n!$$

The Gamma function is an extension of the normal factorial function. Here are a couple of quick facts for the Gamma function

$$\Gamma(p+1) = p\Gamma(p)$$

$$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$